Student Number:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2013

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11 - 16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

1 Let $z = 1 + i$. What is the value of z^{12}?

(A) 64

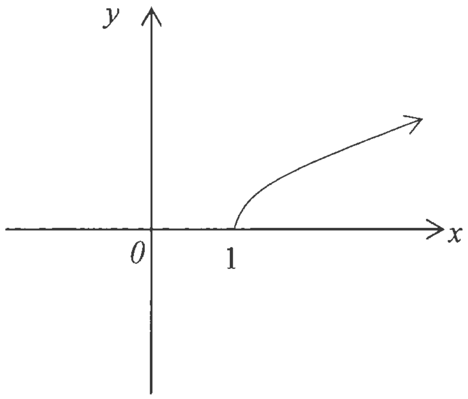
(B) -64

(C) $64i$

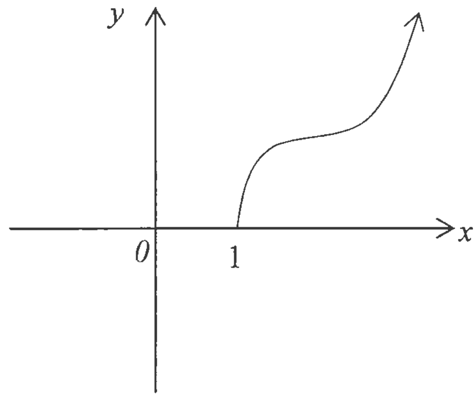
(D) $-64i$

2 Given $f(x) = x^2(x-1)$. Which of the following best represents the graph of $y = \sqrt{f(x)}$?

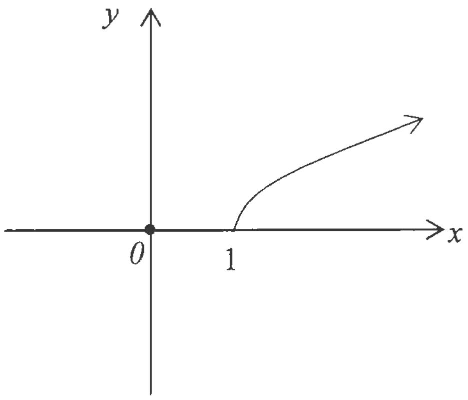
(A)



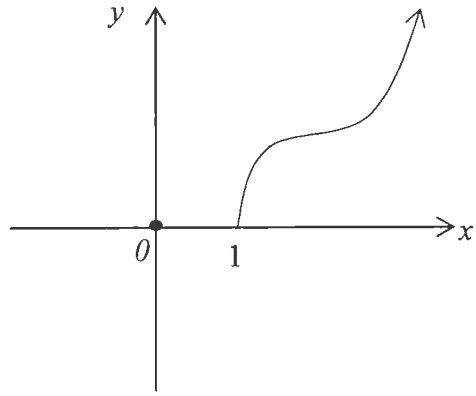
(B)



(C)



(D)



3 Given $2x^2 + xy + 2y^2 = 30$, what are the coordinates of one of the vertical tangents?

(A) (-1, 4)

(B) (4, -1)

(C) (-1, -4)

(D) (1, -4)

4 What is the equation of the chord of contact of tangents from (2, 1) to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$?

(A) $\frac{2x}{9} - \frac{y}{4} = 1$

(B) $\frac{2x}{9} + \frac{y}{4} = 1$

(C) $\frac{x}{9} - \frac{y}{2} = 1$

(D) $\frac{x}{9} + \frac{y}{4} = 1$

5 Given $3x^3 - 2x + 5 = 0$ has roots α, β and γ, what is the equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$?

(A) $3x^3 - 9x^2 + 7x + 6 = 0$

(B) $3x^3 + 9x^2 + 7x + 6 = 0$

(C) $3x^3 - 9x^2 + 7x + 4 = 0$

(D) $3x^3 + 9x^2 + 7x + 4 = 0$

6 Which of the following is the correct expression for the integral $\int \frac{dx}{4+\sin^2 x}$?

(A) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{5}{4}\tan x\right)+C$

(B) $2\sqrt{5}\tan^{-1}\left(\frac{5}{4}\tan x\right)+C$

(C) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}}{2}\tan x\right)+C$

(D) $2\sqrt{5}\tan^{-1}\left(\frac{\sqrt{5}}{2}\tan x\right)+C$

7 Given $3x^3+6x-5=0$ has roots $\alpha,\ \beta$ and γ, what is the value of $\alpha^3+\beta^3+\gamma^3$?

(A) 5

(B) 9

(C) 15

(D) -1

8 The equation of motion of a particle falling with velocity v m/s is given by $\ddot{x}=10-\frac{v}{2}$. Which of the following is the value of the terminal velocity?

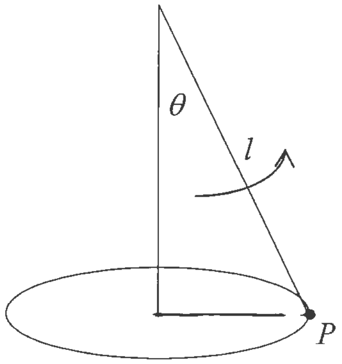
(A) 5

(B) 15

(C) 20

(D) $\sqrt{20}$

9 A bob P of mass m kg is suspended from a fixed point A by a string of length l metres, and acceleration due to gravity g. P describes a horizontal circle with uniform angular velocity ω rad/s.



Which of the following expressions represents the tension in the string?

(A) $ml\omega$

(B) $ml\omega^2$

(C) $mgl\omega$

(D) $mgl\omega^2$

10 Which of the following is the correct expression for the integral $\int e^{\alpha x} \sin \beta x \, dx$?

(A) $\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}[\beta \sin \beta x + \alpha \cos \beta x] + C$

(B) $\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}[\beta \sin \beta x - \alpha \cos \beta x] + C$

(C) $\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}[\alpha \sin \beta x + \beta \cos \beta x] + C$

(D) $\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}[\alpha \sin \beta x - \beta \cos \beta x] + C$

Section II

90 marks
Attempt Questions 11–16.
Allow about 2 hours and 45 minutes for this section.
In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW page.

(a) $|z|<1$ and $z=\cos\theta+i\sin\theta$, where $-\pi<\theta\le\pi$..

(i) Show $1+z=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$. 2

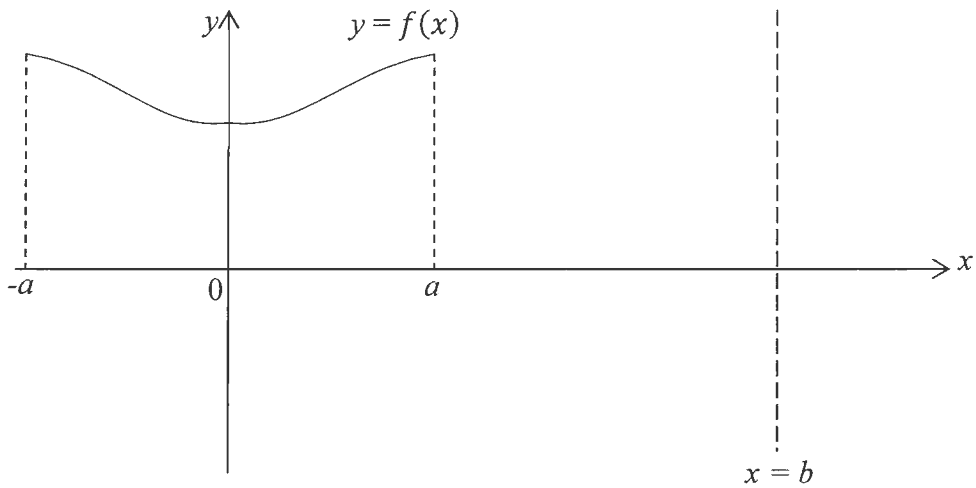
(ii) z_1 and z_2 are complex numbers such that $|z_1|=|z_2|=1$. If z_1 and z_2 have arguments α and β respectively, where $-\pi<\alpha\le\pi$ and $-\pi<\beta\le\pi$, show that $\frac{z_1+z_1z_2}{z_1+1}$ has modulus $\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ and Argument $\frac{\alpha+\beta}{2}$. 2

(iii) If $|z_1|=|z_2|=1$ and $\frac{z_1+z_1z_2}{z_1+1}=2i$ find z_1 and z_2 in the form $x+iy$ where x and y are real rational numbers. 4

(b) Shade the region $-\frac{\pi}{4}\le\operatorname{Arg} z\le\frac{\pi}{4}$ and $|z|\le 3$. 2

Question 11 (c) is continued over the page.

(c)

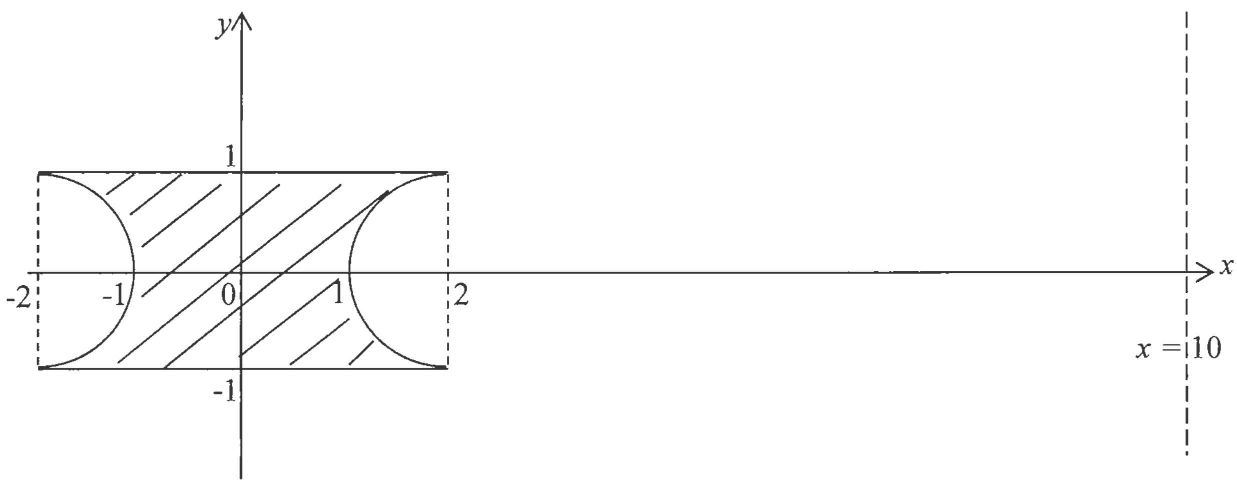


$f(x)$ is an even function such that $f(x) \geq 0$ for $-a \leq x \leq a$.

The region bounded by $y = f(x)$, the x-axis, and the ordinates $x = -a$ and $x = a$ has area A. The region is rotated about the line $x = b$ where $b > a > 0$.

(i) Using the method of cylindrical shells show that the volume V of rotation is $2\pi bA$. 3

(ii)



The region shown with circular ends is rotated about $x = 10$ to form a circular sealing ring. Find the volume of revolution. 2

End of Question 11.

Question 12 (15 marks) Start a NEW page.

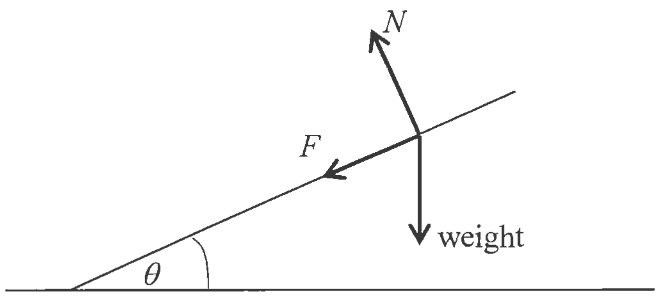
(a) Graph $y = \dfrac{x}{(x+4)(x+2)}$ showing all intercepts with the coordinate axes and all 3
asymptotes.

(b) The region bounded by $y = \dfrac{x}{(x+4)(x+2)}$, the x-axis and $x = 1$ is rotated around the y-axis.

(i) Find the values A, B and C such that $\dfrac{x^2}{(x+4)(x+2)} = A + \dfrac{B}{x+2} + \dfrac{C}{x+4}$. 4

(ii) Using the method of cylindrical shells show that the volume V of revolution is 4
given by $V = 2\pi\displaystyle\int_0^1 \frac{x^2\,dx}{(x+4)(x+2)}$, hence find the exact value of the volume of revolution.

(c)



A car of mass 2000 kg travels around a curve of radius 150 m at a speed of 110km/h. 4
The car experiences a lateral resistance force F of 0.22 × normal force, N, as shown.

By resolving the forces vertically and horizontally find the ~~minimum~~ angle θ ~~(to the nearest minute)~~ for the car to negotiate the curve. (Assume acceleration due to gravity of 10 m/s^2).

End of Question 12.

Question 13 (15 marks) Start a NEW page.

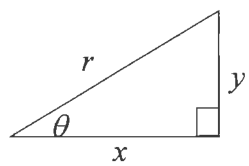
(a) (i) Show $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-x)dx$ 1

(ii) Deduce $\int_{-a}^{a} f(x)dx = \int_{0}^{a} \left[f(x)+f(-x)\right] dx$ 1

(iii) Hence evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+\sin x)^2}$ 4

(b) A shape is defined as $r = \dfrac{9}{5+4\cos\theta}$ where r is the distance from origin and θ is the angle anticlockwise from the positive x-axis.

(i) Using the notation 3



find the equivalent Cartesian equation and show that the shape is an ellipse translated.

(ii) State the minor axis, major axis and location of the foci. 4

(iii) The area A enclosed by the shape is given by $A = \int_{0}^{2\pi} \frac{1}{2} r^2 d\theta$. 2

Using (b)(i) and(b)(ii) evaluate $\displaystyle\int_{0}^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2}$.

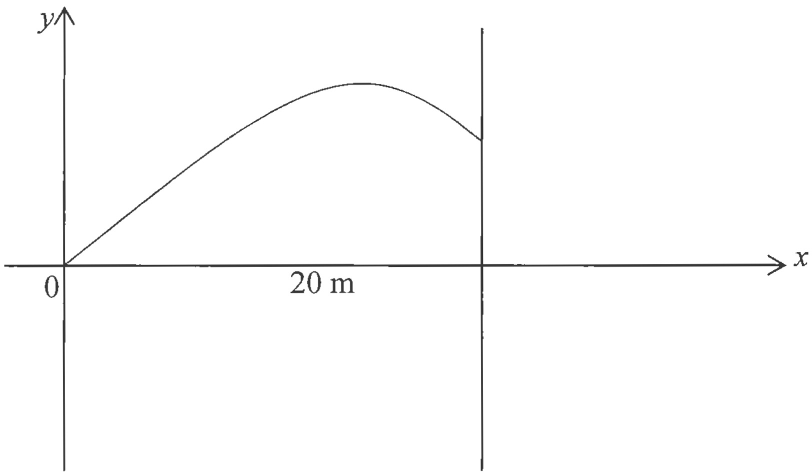
End of Question 13.

Question 14 (15 marks) Start a NEW page.

(a) (i) Find the coordinates of the intersection of the curves $y^2 = 8x$ and $x^2 = 8y$. 1

(ii) The base of a solid is in the region bounded by the curves $y^2 = 8x$ and $x^2 = 8y$, and its cross sections by planes perpendicular to the x-axis are semicircles. Find the volume of the solid. 3

(b)



A liquid particle of mass m kg is projected from the ground and hits a vertical wall 20m from the point of projection as shown.

(i) The equations of motion before the particle hits the wall are 3

$x = 4t$ and $y = 30t - 5t^2$

where t is time in seconds. Show that the particle hits the wall 25 m above the ground with a downwards velocity of 20 m/s.

(ii) After hitting the wall the particle slides down the wall with a resistance force equal to $0.04mv^2$.

(α) If acceleration due to gravity is 10 m/s^2 show that the velocity on return to the ground is approximately 16.44 m/s. 4

(β) Find the total time for the particle to return to the ground. Give your answer to two decimal places. 4

End of Question 14.

Question 15 (15 marks) Start a NEW page.

The hyperbola $xy = c^2$ meets the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at $P\left(ct_1, \dfrac{c}{t_1}\right)$ and $Q\left(ct_2, \dfrac{c}{t_2}\right)$ where $t_1 > t_2 > 0$. Tangents to the hyperbola at *P* and *Q* meet at *T*, while tangents to the ellipse at *P* and *Q* meet at *V*.

(i) Show the above information on a sketch. 1

(ii) Show that the parameter of point $\left(ct, \dfrac{c}{t}\right)$ which lies on the intersection of $xy = c^2$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ satisfies the equation $b^2c^2t^4 - a^2b^2t^2 + a^2c^2 = 0$. 2

(iii) Given the equation of the tangent to the hyperbola at $\left(ct, \dfrac{c}{t}\right)$ is $x + t^2 y = 2ct$, show that the coordinates of *T* are $\left(\dfrac{2ct_1t_2}{t_1+t_2}, \dfrac{2c}{t_1+t_2}\right)$. 2

(iv) Given that the equation of the tangent to the ellipse at (x_1, y_1) is $b^2x_1x + a^2y_1y = a^2b^2$, show that the coordinates of *V* are $\left(\dfrac{a^2}{c(t_1+t_2)}, \dfrac{b^2t_1t_2}{c(t_1+t_2)}\right)$. 2

(v) Show that the line *TV* passes through the origin. 3

(vi) Point *V* lies at a focus of the hyperbola.

(α) Show that the ellipse is a circle. 2

(β) Find the radius of the circle in terms of *c*. 3

End of Question 15.

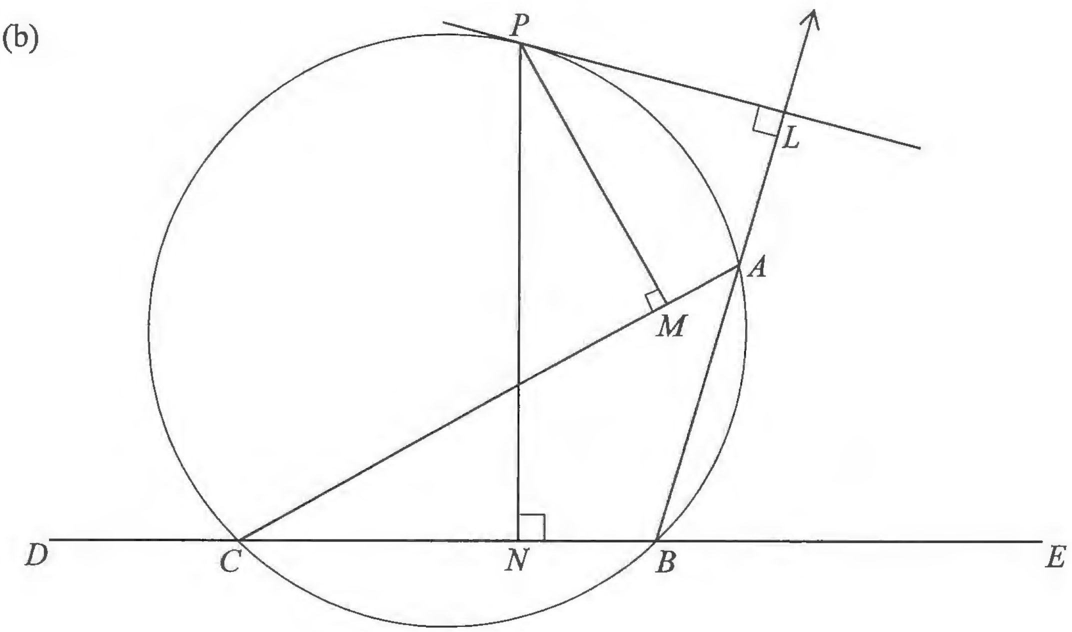
Question 16 (15 marks) Start a NEW page.

(a) $I_n = \int_0^{2\pi} (1+\cos\theta)^n \, d\theta$ for $n \geq 0$.

(i) Show $I_{n+1} = \dfrac{2n+1}{n+1} I_n$. 4

(ii) Find I_3. 1

(b)



ABC is a triangle inscribed in a circle. L, M and N are the feet of the perpendiculars from P to AB, AC and BC respectively.

(i) Copy the diagram. 1

(ii) Show P, M, A and L are concyclic points. 2

(iii) Show P, C, N and M are concyclic points. 2

(iv) Show that L, M and N are collinear. 5

End of paper.

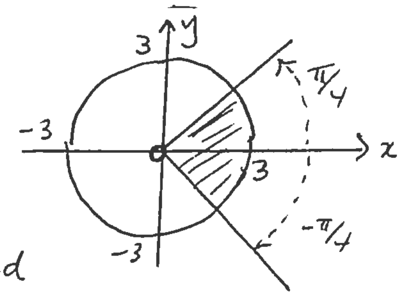
» Section I

1 mk for each question.

1. A
2. D
3. B
4. A
5. C
6. C
7. A
8. C
9. B
10. D

2013 TRIAL **MATHEMATICS Extension 2: Question** 11

Suggested Solutions	Marks	Marker's Comments
(a)(i) $1+z=(1+\cos\theta)+i(\sin\theta)$ $=(1+\cos 2\times\frac{\theta}{2})+i(\sin 2\times\frac{\theta}{2})$ $=2\cos^2\frac{\theta}{2}+i(2\sin\frac{\theta}{2}\cos\frac{\theta}{2})$ $=2\cos\frac{\theta}{2}(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})$ or $2\cos\frac{\theta}{2}\,\mathrm{cis}\frac{\theta}{2}$	1 1	This part was well done by most students
(ii) $z_1=2\cos\frac{\alpha}{2}\,\mathrm{cis}\frac{\alpha}{2}$ $z_2=2\cos\frac{\beta}{2}\,\mathrm{cis}\frac{\beta}{2}$ $\left\lvert\frac{z_1(1+z_2)}{1+z_1}\right\rvert=\frac{\lvert z\rvert\,\lvert 1+z_2\rvert}{\lvert 1+z_1\rvert}$ $=\frac{(1)(2\cos\frac{\beta}{2})}{(2\cos\frac{\alpha}{2})}$ $=\frac{\cos\beta/2}{\cos\alpha/2}$ NOTE This is positive since $-\pi<\alpha,\beta<\pi$ $\arg\left(\frac{z_1(1+z_2)}{1+z_1}\right)=\arg z_1+\arg(1+z_2)-\arg(1+z_1)$ $=\alpha+\beta/2-\alpha/2$ $=\frac{\alpha+\beta}{2}$	1 1	Quite a few failed to factorise $z_1+z_1z_2$ and hence did not use a(i) which made the question more difficult. A significant number of students confused $\cos\frac{\alpha}{2}$ and $\mathrm{cis}\frac{\alpha}{2}$

MATHEMATICS Extension 2: Question 11		
Suggested Solutions	**Marks**	**Marker's Comments**
(iii) $\arg(2i) = \frac{\pi}{2}$ $\therefore \frac{\alpha+\beta}{2} = \frac{\pi}{2}$ $\alpha+\beta = \pi$	1	Some thought that the $\arg 2i = \pi$
$\lvert 2i \rvert = 2$ $\frac{\cos \beta/2}{\cos \alpha/2} = 2$ $\frac{\cos(\pi/2 - \alpha/2)}{\cos \alpha/2} = \frac{\sin \frac{\alpha}{2}}{\cos \alpha/2} = 2$ $t = \tan \alpha/2 = 2$	1	Many missed the fact that $\cos(\pi/2 - \alpha/2) = \sin \alpha/2$ This mark for $\tan \alpha/2 = 2$ or $\tan \beta/2 = 1/2$
$\cos\alpha = \frac{1-t^2}{1+t^2} = \frac{1-2^2}{1+2^2} = -3/5$ $\sin\alpha = \frac{2t}{1+t^2} = \frac{2\times 2}{1+2^2} = 4/5$ $\therefore z_1 = -3/5 + 4/5\, i$	1	Many made arithmetic mistakes or assumed things like $z_1 = -z_2$ or $z_1 = z_2$
Similarly $\tan \beta/2 = 1/2$ $\cos\beta = 3/5 \quad \sin\beta = 4/5$ $z_2 = \frac{3}{5} + 4/5\, i$	1	Full marks for z_1 & z_2 correctly obtained.
(b)  NOTE (0,0) excluded	1 1	Some students did not note the $\pi/4$ angles. Quite a few did not note that the origin is excluded

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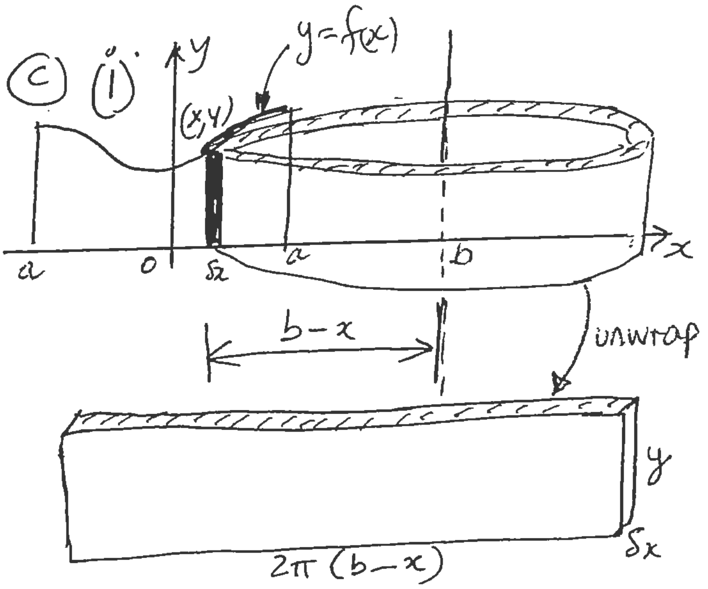
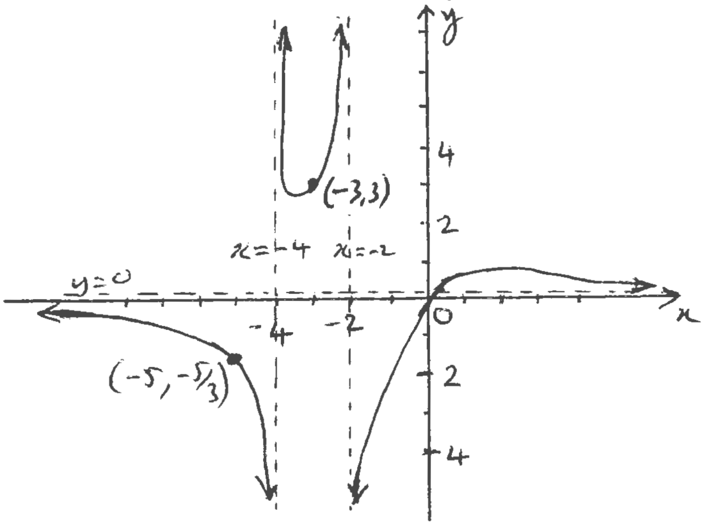
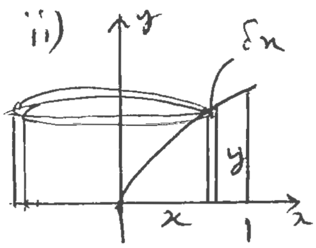
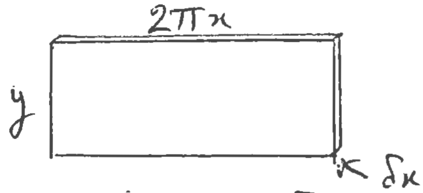
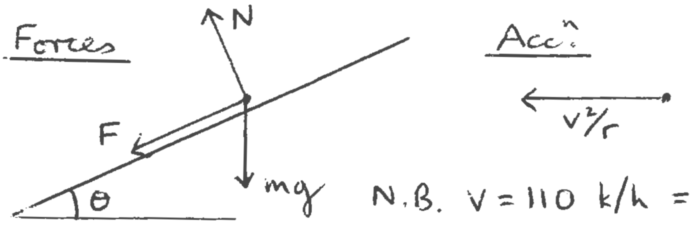
MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments
(c) (i) 		
Volume $= \lim_{\delta x \to 0} 2\pi(b-x)y\,\delta x$ $V = \int_{-a}^{a} 2\pi f(x)(b-x)\,dx$ *	1	Some students stated * without justification and were not awarded full marks
$V = 2\pi b \int_{-a}^{a} f(x)\,dx - 2\pi\int_{-a}^{a} x f(x)\,dx$	1	Students needed to explain why $\int_{-a}^{a} f(x)dx = A$ and $\int_{-a}^{a} xf(x)dx = 0$
$V = 2\pi b A$ since $\int_{-a}^{a} f(x) = A$ and $\int_{-a}^{a} x f(x) = 0$ since $x f(x)$ is odd (odd func. x even func = odd func.)	1	
(ii) From (i) $V = 2\pi b A$ $A =$ rectangle − circle $= 4\times 2 - \pi(1)^2$ $= 8 - \pi$ $b = 10$	1	Many students wasted time by not using c(i) but by finding the volume by integration.
$V = 2\pi(10)(8-\pi)$ $V = 20\pi(8-\pi)$ units^3	1	A common error was to think that the area of the rectangle was 4

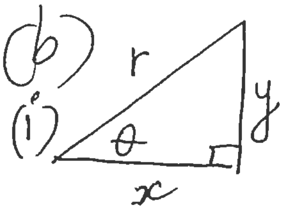
Suggested Solutions	Marks	Marker's Comments
a) Vert Asy $x=-4,\ x=-2$ Hor. Asy $y=0$ Zero at $x=0$, also y intercept.	$3\times\frac{1}{2}$	for each asymptote with either an equation or a line definitely finishing towards it.
	1	for shape - ½ off for each real error in main graph.
	$\frac{1}{2}$	for having a labelled point on each branch. (or associated scales)
b) i) $\dfrac{x^2}{(x+4)(x+2)} = A + \dfrac{B}{x+2} + \dfrac{C}{x+4}$		
$\therefore A(x+2)(x+4) + B(x+4) + C(x+2) \equiv x^2$	1	Easy marks.
Equate coeffs. of x^2 : $\underline{\underline{A=1}}$	1	
Put $x=-2$: $2B=4$ $\therefore \underline{\underline{B=2}}$	1	
Put $x=-4$: $-2C=16$ $\therefore \underline{\underline{C=-8}}$	1	
ii)  	$\frac{1}{2}$	diagram
$\delta V \doteqdot 2\pi x y\,\delta x$ $\delta V = \left(\pi(x+\delta x)^2 - \pi x^2\right) y$	$\frac{1}{2}$	for $\doteqdot$ (type 2) or neglect 2nd order terms
$= 2\pi x y\,\delta x$ (neglecting 2nd order terms)	$\frac{1}{2}$	for limit of sum
$\therefore V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{1} 2\pi x y\,\delta x$ $= 2\pi \displaystyle\int_0^1 \frac{x^2\,dx}{(x+2)(x+4)}$	$\frac{1}{2}$	for integral (except if baldly stated)

Suggested Solutions	Marks	Marker's Comments
b) ii) (cont)		
$\therefore V_\Omega = 2\pi\int_0^1 1 + \frac{2}{x+2} - \frac{8}{x+4}\,dx$ (using part i)		
$= 2\pi\left[x + 2\ln(x+2) - 8\ln(x+4)\right]_0^1$	1	
$= 2\pi\{1 + 2\ln 3 - 8\ln 5 - 2\ln 2 + 8\ln 4\}$	1	
$\underline{\underline{V_\Omega = 2\pi\{1 + 2\ln 3 + 14\ln 2 - 8\ln 5\}\ u^3}}$		
c) Forces Acc^n  N.B. $V = 110$ k/h $= \frac{275}{9}$ m/s	½	Most people got these 2½ marks.
Resolve vertically (V) $mg + F\sin\theta = N\cos\theta$	1	
Resolve horizontally (H) $F\cos\theta + N\sin\theta = \frac{mv^2}{r}$	1	
(Assuming $F = 0.22N$ means this is already the optimal angle θ.)		
Substituting numbers		
(V) $\rightarrow$ $N(\cos\theta - 0.22\sin\theta) = 20000$		Many mistakes in the numeric work. ½ for getting down to a simple equation in $\tan\theta$.
(H) $\rightarrow$ $N(0.22\cos\theta + \sin\theta) = 12448.56$		
Dividing: $\frac{\cos\theta - 0.22\sin\theta}{0.22\cos\theta + \sin\theta} = 1.6066\ldots$		
$\frac{1 - 0.22\tan\theta}{0.22 + \tan\theta} = 1.6066\ldots$		
$\tan\theta = 0.3539$	½	Final mark for correct solution
$\underline{\underline{\theta = 19^\circ 29'}}$ (nearest minute)	1	(29' or 30' accepted)

MATHEMATICS Extension 2: Question.

Suggested Solutions	Marks	Marker's Comments
(a) (i) Let $\left.\begin{array}{l} x=-u \\ dx=-du \\ x=0 \quad u=0 \\ x=-a \quad u=a \end{array}\right\}$ $\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du)$ $= \int_{0}^{a} f(-u)\,du$ $= \int_{0}^{a} f(-x)\,dx$ Changing the variable in a definite integral does not change its value	1	Well done by students Some students thought that the function must be even (or must be odd)
(ii) $\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ $= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$ $= \int_{0}^{a} [f(x) + f(-x)]\,dx$	1	Well done by students
(iii) $\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+\sin x)^2} = \int_{0}^{\pi/4} \frac{1}{(1+\sin x)^2} + \frac{1}{[1+\sin(-x)]^2}\,dx$ $= \int_{0}^{\pi/4} \frac{1}{(1+\sin x)^2} + \frac{1}{(1-\sin x)^2}\,dx$ $= \int_{0}^{\pi/4} \frac{(1-\sin x)^2 + (1+\sin x)^2}{(1-\sin^2 x)^2}\,dx$	1	Nearly all students used a(ii) correctly to begin

MATHEMATICS Extension 2: Question..

Suggested Solutions	Marks	Marker's Comments
$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+\sin x)^2} = \int_0^{\pi/4} \frac{2(1+\sin^2 x)}{\cos^4 x}\,dx \quad (A)$	1	Most sudents got to (A).
$= 2\int_0^{\pi/4} \sec^2 x(\sec^2 x+\tan^2 x)\,dx$		Many failed to realise $\int \sec^2 x \tan^2 x\,dx = \frac{1}{3}\tan^3 x$
$= 2\int_0^{\pi/4} \sec^2 x(1+2\tan^2 x)\,dx$	1	
$= 2\left[\tan x + \frac{2}{3}\tan^3 x\right]_0^{\pi/4}$		
$= 10/3$	1	Correct answer correctly done (by many of a variety of methods) for full marks
(b) (i)  $r = \sqrt{x^2+y^2}$ $\cos\theta = \frac{x}{r}$		
$r = \frac{9}{5+4\cos\theta}$		
$r = \frac{9}{5+4\left(\frac{x}{r}\right)}$		
$1 = \frac{9}{5r+4x}$		Most students failed to eliminate both θ and r so we unable to make progress
$5r = 9-4x$		
$25r^2 = 81-72x+16x^2$		
$25(x^2+y^2) = 81-72x+16x^2$	1	
$9x^2+72x+25y^2 = 81$		
$9(x^2+8x+16)+25y^2 = 81+9\times16$	1	Arithmetic mistakes were common here.
$9(x+4)^2+25y^2 = 225$		
$\frac{(x+4)^2}{25}+\frac{y^2}{9} = 1$	1	Complete simplication required for full marks.
This is an ellipse, centre $(-4,0)$		

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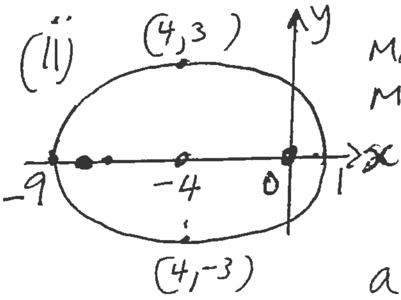
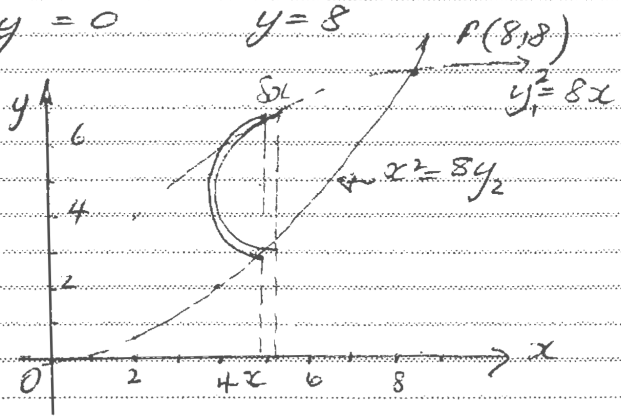
Suggested Solutions	Marks	Marker's Comments
(ii) 		Many students confused semiaxis (a or b) with axis (2a or 2b)
MAJOR AXIS $= 2 \times 5 = 10$ units	1	
MINOR AXIS $= 2 \times 3 = 6$ units	1	
$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{4}{5}$		
$ae = 4 \times \frac{4}{5} = 4$	1	
FOCI: $(-4 \pm 4, 0) \Rightarrow (-8, 0)$ and $(0, 0)$	1	
(iii) AREA $= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$ $\pi ab = \int_0^{2\pi} \frac{1}{2} \left(\frac{9}{5 + 4\cos\theta}\right)^2 d\theta$		Many wasted time finding the area of the ellipse by integration instead of quoting $A = \pi ab$
Area $= \pi \times 3 \times 5$ $= 15\pi$	1	
$\therefore \int_0^{2\pi} \frac{d\theta}{(5 + 4\cos\theta)^2}$ $= 15\pi \times \frac{2}{81}$		
$= \frac{10\pi}{27}$	1	Full marks for correct answer correctly obtained.

Suggested Solutions	Marks	Marker's Comments
14 a) (i) Find the points of intersection of $x^2=8y$ and $y^2=8x$		
$x^4=(8y)^2$		
$x^4=64\times 8x$		
$x^4=512x$		
$x^4-512x=0$		
$x(x^3-512)=0$		
$x(x-8)(x^2+8x+64)=0$		
$x=0$ or $x=8$ or no solution		
$\therefore y=0$ $\quad y=8$	1	
(ii) 		
Area of Cross Section $=\frac{1}{2}\left(\pi\frac{D^2}{4}\right)$		
$=\frac{\pi}{8}(y_1-y_2)^2$		
$=\frac{\pi}{8}\left(2\sqrt{2x}-\frac{x^2}{8}\right)^2$		
Volume of Slice $=A.\delta x$		
Volume $=\lim\limits_{\delta x\to 0}\sum\limits_{0}^{8}A(x)\cdot\delta x$		
$\therefore V=\frac{\pi}{8}\int_0^8\left(2\sqrt{2}x-\frac{x^2}{8}\right)^2$		
$=\frac{\pi}{8}\int_0^8 8x-\frac{\sqrt{2}}{2}x^{5/2}+\frac{x^4}{64}\,dx$		
$=\frac{\pi}{8}\left[4x^2-\frac{\sqrt{2}}{7}x^{7/2}+\frac{x^5}{320}\right]_0^8$		
$=\frac{\pi}{8}\left[4(8)^2-\frac{\sqrt{2}}{7}(8)^{7/2}+\frac{8^5}{320}-0\right]$		
$=32\pi\left[1-8/7+2/5\right]$		
$\therefore V=\frac{288\pi}{35}$ units^3.		

Suggested Solutions	Marks	Marker's Comments

14 b) Given $x = 4t$ and $y = 30t - 5t^2$
the particle hits the wall when
$x = 20\,m$ and $y = 25\,m$

(i) $4t = 20$
$t = 5$

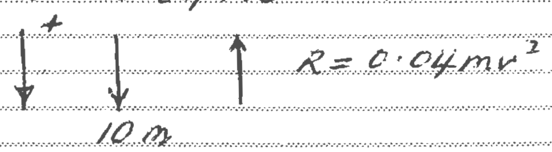
$y = 30 \times 5 - 5(5)^2$
$= 150 - 125$
$= 25$ 3

$\therefore$ The particle does hit the wall 25 m above the ground.

And, $y' = 30 - 10t$
$= 30 - 10 \times 5$
$= {}^-20$

$\therefore$ The particle has a downwards velocity of 20 m/s.

(ii) After hitting the wall, the particle slides down with a resistance force of $0.04mv^2$



Sum of the forces $= m\ddot{x}$ (Newtons 2nd Law)

ie $m\ddot{x} = 10m - 0.04mv^2$
$\ddot{x} = 10 - 0.04v^2$
$\therefore \ddot{x} = -0.04(v^2 - 250)$ $\quad v > \sqrt{250}$

(α) For velocity on return to the ground 4

$v \cdot \frac{dv}{dx} = -0.04(v^2 - 250)$

$$\int_{20}^{v} \frac{v\,dv}{v^2 - 250} = \int_0^{25} -0.04\,dx$$

$$\left[\tfrac{1}{2}\ln(v^2 - 250)\right]_{20}^{v} = -0.04\left[x\right]_0^{25}$$

$$\tfrac{1}{2}\left[\ln(v^2 - 250) - \ln(20^2 - 250)\right] = -0.04[25 - 0]$$

$$\tfrac{1}{2}\ln\left(\frac{v^2 - 250}{150}\right) = -1$$

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Suggested Solutions	Marks	Marker's Comments

14 (b)(ii) (α) continued..

$$\ln\left(\frac{v^2-250}{150}\right) = -2$$

$$\frac{v^2-250}{150} = e^{-2} \qquad \text{(take exponentials of both sides)}$$

$$v^2 = 150e^{-2}+250$$

$$= 270\cdot30029\ldots$$

$$v = 16\cdot4408\ldots$$

$\therefore$ The velocity on return to the ground is approximately $16\cdot44$ m/s

(β) Find the total time for the particle to return to the ground.

$$\ddot{x} = -0\cdot04(v^2-250) \quad \text{from (i)}$$

$$\text{ie } \frac{dv}{dt} = -0\cdot04(v^2-250)$$

$$\int_{20}^{16\cdot44}\frac{dv}{v^2-250} = \int_0^T -0\cdot04\,dt$$

NB
$$\frac{1}{v^2-250} = \frac{A}{v-\sqrt{250}} + \frac{B}{v+\sqrt{250}}$$

$$1 = A(v+\sqrt{250}) + B(v-\sqrt{250})$$

$$1 = v(A+B) + \sqrt{250}(A-B)$$

$$A-B = \frac{1}{\sqrt{250}} \qquad A+B=0$$

$$A = -B$$

$$2A = \frac{1}{\sqrt{250}} \Rightarrow A = \frac{1}{2\sqrt{250}} \;\&\; B = \frac{-1}{2\sqrt{250}}$$

NOW,

$$\frac{1}{2\sqrt{250}}\int_{20}^{16\cdot44}\left(\frac{1}{v-\sqrt{250}} - \frac{1}{v+\sqrt{250}}\right)dt = -0\cdot04\Big[t\Big]_0^T$$

$$\frac{1}{2\sqrt{250}}\left[\ln\left(\frac{v-\sqrt{250}}{v+\sqrt{250}}\right)\right]_{20}^{16\cdot44} = -0\cdot04\left[T-0\right]$$

$$T = \frac{-25}{2\sqrt{250}}\ln\left(\frac{16\cdot44-\sqrt{250}}{16\cdot44+\sqrt{250}} \times \frac{20+\sqrt{250}}{20-\sqrt{250}}\right)$$

$$= -0\cdot79056 \times \ln\left(\frac{0\cdot628611\ldots}{32\cdot25138\ldots} \times \frac{35\cdot81138\ldots}{4\cdot18861\ldots}\right)$$

$$= 1\cdot41662\ldots$$

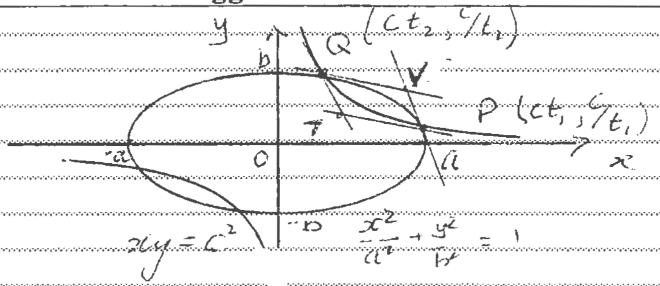
Total Time $= 1\cdot42 + 5$

$= 6\cdot42$ seconds (to 2 d.p.)

Marks: 4

(1)

MATHEMATICS Extension 2: Question 15

Suggested Solutions	Marks	Marker's Comments
(i) 	1	$\frac{1}{2}$ for correct position of P and Q $\frac{1}{2}$ for V and T
(ii) The point $(ct, \frac{c}{t})$ lies on $xy = c^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{(ct)^2}{a^2} + \frac{(c/t)^2}{b^2} = 1$ $b^2c^2t^2 + a^2c^2 = a^2b^2t^2$ $b^2c^2t^2 + a^2c^2 - a^2b^2t^2 = 0$	2	1 sub $(ct, c/t)$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 1 simplifying
(iii) Equation of tangent is $x + t^2y = 2ct$ at P : $x + t_1^2 y = 2ct_1$ (i) at Q : $x + t_2^2 y = 2ct_2$ (ii) $y(t_1^2 - t_2^2) = 2c(t_1 - t_2)$ $y(t_1 - t_2)(t_1 + t_2) = 2c(t_1 - t_2)$ $y = \frac{2c}{t_1 + t_2}$ sub into (i) $x = 2ct_1 - t_1^2\left[\frac{2c}{t_1+t_2}\right]$ $= \frac{2c\left[t_1^2 + t_1t_2 - t_1^2\right]}{t_1+t_2}$ $x = \frac{2ct_1t_2}{t_1+t_2}$ $T = \left[\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right]$	2 $t_1 \neq t_2$	no loss of mark if $t_1 \neq t_2$ not written 1 x coordinate 1 y coordinate
(ii) Equation of tangent is $b^2x_1x + a^2y_1y = a^2b^2$ at P : $bct_1x + \frac{a^2c}{t_1}y = a^2b^2 \quad \times t_2$ at Q $bct_2x + \frac{ac}{t_2}y = a^2b^2 \quad \times t_1$ (3) $bct_1t_2x + a^2c\frac{t_2}{t_1}y = a^2b^2t_2$ (4) $bct_1t_2x + a^2c\frac{t_1}{t_2}y = a^2b^2t_1$ (3) − (4) $a^2yy\left[\frac{t_2}{t_1} - \frac{t_1}{t_2}\right] = a^2b^2\left[t_2 - t_1\right]$ $a^2c\left[\frac{t_2^2 - t_1^2}{t_1t_2}\right] = a^2b^2\left[t_2 - t_1\right]$		

MATHEMATICS Extension 2 : Question 15

Suggested Solutions	Marks	Marker's Comments

$$y = \frac{b^2 t_1 t_2 (t_2 - t_1)}{c(t_1 - t_2)(t_1 + t_2)} \qquad t_1 \neq t_2$$

Marks: 2

$$= \frac{b^2 t_1 t_2}{c(t_1+t_2)}$$

1 x coordinate

1 y coordinate

(i) $\times t_1$ ⑤ $b^2 c t_1^2 x + a^2 c y = a^2 b^2 t_1$

(ii) $\times t_2$ ⑥ $b^2 c t_2^2 x + a^2 c y = a^2 b^2 t_2$

⑤ − ⑥

$$b^2 c x [t_1^2 - t_2^2] = a^2 b^2 [t_1 - t_2]$$

$$x = \frac{a^2 (t_1 - t_2)}{c(t_1 - t_2)(t_1 + t_2)} = \frac{a^2}{c(t_1+t_2)}$$

$$V = \left[\frac{a^2}{c(t_1+t_2)}, \frac{b^2 t_1 t_2}{c(t_1+t_2)}\right]$$

(v) Gradient of OT $\quad m_{OT} = \frac{2c}{t_1+t_2} \Big/ \frac{2c t_1 t_2}{t_1 + t_2}$

$$= \frac{1}{t_1 t_2}$$

Gradient of OV $\quad m_{OV} = \frac{b^2 t_1 t_2}{c(t_1+t_2)} \Big/ \frac{a^2}{c(t_1+t_2)}$

$$= \frac{b^2}{a^2}\left[t_1 t_2\right]$$

1 gradients OT, OV

Roots of $b^2 c^2 t^4 - a^2 b^2 t^2 + a^2 c^2 = 0$

t_1, t_2 and $-t_1, -t_2$ by symmetry

product of roots $t_1^2 t_2^2 = \frac{a^2 c^2}{b^2 c^2}$

$\therefore t_1 t_2 = a/b$ as $t_1 t_2 > 0$

1 $t_1 t_2 = a/b$.

$$\therefore m_{OV} = \frac{b^2}{a^2} \times a/b = b/a$$

$$m_{OT} = \frac{1}{t_1 t_2} = b/a$$

$\therefore$ V, O, T collinear (2 equal gradients and common point)

$\therefore$ VT passes through origin

1 Conclusion with working

Alternatively Equation of TV

$$y - \frac{2c}{t_1+t_2} = \frac{b^2 t_1 t_2 - 2c^2}{a^2 - 2c^2 t_1 t_2}\left[x - \frac{2c\, t_1 t_2}{t_1+t_2}\right]$$

LHS = RHS when $x = 0$ $y = 0$ and $t_1 t_2 = a/b$

$\therefore$ TV passes through origin

Alternatively

1 Gradient TV

1 Equation of TV and sub (0, 0)

1 showing correctly LHS = RHS (using $t_1 t_2 = a/b$)

MATHEMATICS Extension 2 : Question 15

Suggested Solutions	Marks	Marker's Comments
(vi) (α) Focus $= V(c\sqrt{2}, c\sqrt{2})$	2	
$x_v = \dfrac{a^2}{c(t_1+t_2)} = c\sqrt{2}$		
$y_v = \dfrac{b^2 t_1 t_2}{c(t_1+t_2)} = c\sqrt{2}$		① relating a and b
$\therefore \dfrac{x_v}{y_v} = \dfrac{a^2}{b^2 t_1 t_2} = 1$		
$a^2 = b^2 t_1 t_2 \qquad t_1 t_2 = \frac{a}{b}$		
$a^2 = b^2 \frac{a}{b} \qquad a^2 = ab$		① Showing $a = b$ (with proof)
$\therefore \dfrac{a}{b} = 1 \qquad a \neq 0\ b \neq 0$		
$a = b$.		
Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad \therefore \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1$		
$\therefore x^2 + y^2 = a^2$		
circle centre $(0,0)$ radius a units	3	
(β) Focus lies on tangent to ellipse		Other methods possible.
$b^2 x x_1 + a^2 y_1 y = a^2 b^2$ (at x_1, y_1)		
$a^2 x x_1 + a^2 y_1 y = a^4 \qquad a = b$		
$x x_1 + y y_1 = a^2$		
Tangent passes through focus $(c\sqrt{2}, c\sqrt{2})$		
$c\sqrt{2} x_1 + c\sqrt{2} y_1 = a^2$		① expression for $x_1 + y_1$
$x_1 + y_1 = \dfrac{a^2}{2c\sqrt{2}}$		
But $x_1^2 + y_1^2 = a^2$ (circle).		
$x_1^2 + 2x_1y_1 + y_1^2 = a^2 + 2x_1y_1$		
$(x_1+y_1)^2 = a^2 + 2c^2$ as $x_1 y_1$		is on $xy = c^2$
$\dfrac{a^4}{2c^2} = a^2 + 2c^2$		
$a^4 = 2c^2a^2 + 4c^4$		① Quadratic Equation in a^2
$a^4 - 2c^2a^2 - 4c^4 = 0$		
$a^2 = \dfrac{2c^2 \pm \sqrt{4c^4 + 16c^4}}{2}$		
$a^2 > 0 \qquad a^2 = \dfrac{2c^2 + c^2\sqrt{20}}{2} = c^2 + c^2\sqrt{5}$		
$a > 0$, $c > 0 \qquad a = c\sqrt{1+\sqrt{5}}$		① Solution

Suggested Solutions	Marks	Marker's Comments

16 a) $I_n = \int_0^{2\pi} (1+\cos\theta)^n \, d\theta \; , \; n \geq 0$

(i) Show $I_{n+1} = \frac{2n+1}{n+1} I_n$

$$I_{n+1} = \int_0^{2\pi} (1+\cos\theta)^{n+1} \, d\theta$$

$$= \int_0^{2\pi} (1+\cos\theta)(1+\cos\theta)^n \, d\theta$$

$$= \int_0^{2\pi} (1+\cos\theta)^n d\theta + \int_0^{2\pi} \cos\theta (1+\cos\theta)^n d\theta$$

Integrating by parts:

$u = (1+\cos\theta)^n \qquad v' = \cos\theta$

$u' = -n(1+\cos\theta)^{n-1} \cdot \sin\theta \qquad v = \sin\theta$

$$I_{n+1} = I_n + \left\{ \Big[(1+\cos\theta)^n \sin\theta\Big]_0^{2\pi} + n\int_0^{2\pi} \sin^2\theta (1+\cos\theta)^{n-1} d\theta \right\}$$

$$= I_n + \left[(1+\cos 2\pi)^n \sin 2\pi - (1+\cos 0)^n \sin 0\right]$$

$$+ n\int_0^{2\pi} (1-\cos^2\theta)(1+\cos\theta)^{n-1} d\theta$$

$$= I_n + 0 - n\int_0^{2\pi} (\cos^2\theta + 2\cos\theta - 2\cos\theta + 2 - 2 - 1)(1+\cos\theta)^{n-1} d\theta$$

$$= I_n - n\int_0^{2\pi} \left[(1+\cos\theta)^2 - 2(1+\cos\theta)\right](1+\cos\theta)^{n-1} d\theta$$

$$= I_n - n\int_0^{2\pi} (1+\cos\theta)^{n+1} - 2(1+\cos\theta)^n \, d\theta$$

$I_{n+1} = I_n - n\, I_{n+1} + 2n\, I_n$

$(n+1) I_{n+1} = (2n+1) I_n$

$I_{n+1} = \frac{2n+1}{n+1} I_n$ #

(ii) Find I_3 : $\quad I_0 = \int_0^{2\pi} (1+\cos\theta)^0 d\theta = \int_0^{2\pi} d\theta = 2\pi$

$I_1 = I_0 = 2\pi$

$I_2 = \frac{2+1}{1+1} \cdot 2\pi = 3\pi$

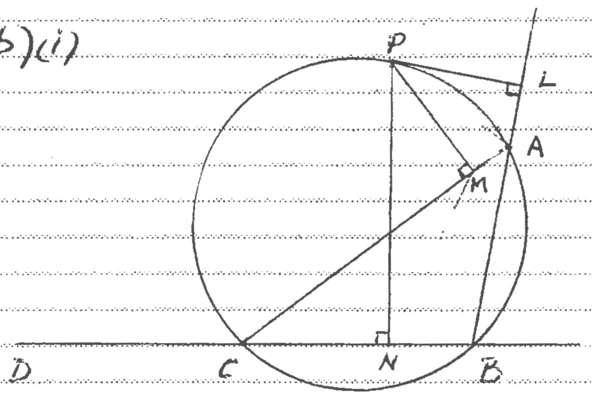
$I_3 = \frac{2(2)+1}{2+1} \cdot 3\pi = 5\pi$

$\therefore I_3 = 5\pi$ #.

MATHEMATICS Extension 2 : Question 16

Suggested Solutions	Marks	Marker's Comments

16(b)(i)



1

(ii) $P\hat{L}A + P\hat{M}A = 90° + 90°$ (L & M are the feet of the perpendiculars from P to AB & AC respectively)
$= 180°$
$\therefore$ PMAL is a cyclic quadrilateral
(opposite angles are supplementary)

$\therefore$ P, M, A and L are concyclic points — 2

(iii) $P\hat{M}C = P\hat{N}C = 90°$ (M & N are the feet of the perpendiculars from P to AC & CB resp'ly)
$\therefore$ PCNM is a cyclic quadrilateral
(angles subtended by interval PC on the same side are equal)
$\therefore$ P, C, N & M are concyclic points — 2

(iv) Show: L, M and N are collinear

Constructions: Join ML, MN, PA & PC

<u>Proof</u>: $P\hat{C}B = P\hat{A}L$ (exterior angle of cyclic quad. PABC equals the interior opposite angle) — 1

$P\hat{A}L = P\hat{M}L$ (angles at the circumference in the same segment of cyclic quad. PMAL) are equal — 1

$\therefore P\hat{C}B = P\hat{M}L$ — 1

Also, $P\hat{C}D = P\hat{M}N$ (exterior angle of cyclic quad. PCMN equals the interior opposite angle) — 1

$P\hat{M}L + P\hat{M}N = P\hat{C}B + P\hat{C}D$
$= 180°$ (straight angle BCD equals 180°)

Now $P\hat{M}L + P\hat{M}N = L\hat{M}N = 180°$
$\therefore L\hat{M}N$ is a straight angle

$\therefore$ L, M & N are collinear — 1